The formal theory of relative monads

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PSSL 107 – Athens 2023

Overview

1. Relative monads
2. Formal category theory
3. The formal theory of relative monads
4. Closing remarks

Relative monads

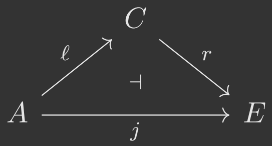
Relative adjunctions

The concept of relative adjunction is a generalisation of the concept of adjunction, where the domain of the left adjoint is permitted to be different to the codomain of the right adjoint.

Definition 1 ([Ulm68])

A relative adjunction comprises

1. a functor $j\colon A \to E$, the *root*;
2. a functor $\ell\colon A \to C$, the *left relative adjoint*;
3. a functor $r\colon C \to E$, the *right relative adjoint*;
4. an isomorphism of the form $C(\ell, 1) \cong E(j, r)$.



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- Partial adjunctions.
- Weighted colimits.
- Nerves.
- Algebraic theories and their various generalisations [Ark22] (cf. [Luc16; BG19]).

Relative monads

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Definition 2 ([ACU10])

A relative monad comprises

1. a functor $j \colon A \to E$, the *root*;
2. a functor $t \colon A \to E$, the *carrier*;
3. a natural transformation $\eta \colon j \Rightarrow t$, the *unit*;
4. a form $\dagger \colon E(j, t) \Rightarrow E(t, t)$, the *extension operator*,

satisfying unitality and associativity axioms.

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- Cocontinuous monads on cocompletions (e.g. finitary monads on locally finitely presentable categories).
- Monads arising from monad–theory correspondences.

Motivation

The theory of *ordinary* relative monads has been substantially developed [Wal70; Die75; ACU15]. However, there are also many motivating examples of relative monads in *enriched* category theory, so we should like an analogous development in this setting.

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Formal category theory

A proliferation of category theories

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- Fibred and indexed category theory.
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In each flavour of category theory, we have essentially the same definitions and theorems.

- Presheaves and the Yoneda lemma.
- Adjoint functor theorems.
- Monadicity theorems.
- Presentability and duality.

⋮

Formal category theory

As category theorists, this situation calls to us for abstraction: if we see essentially the same theorem being reproven again and again in different settings, we should hope that each variant is a consequence of a more general statement.

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Traditionally, this takes the form of applying 2-dimensional category theory to study 1-dimensional category theory.

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An obvious candidate, therefore, is the setting of a 2-category.

Many early approaches to formal category theory took place in the setting of a 2-category equipped with various property-like structure (e.g. limits, colimits, exponentials).

The insufficiency of 2-categories

However, 2-categories turn out to be insufficient to capture many fundamental concepts in (enriched) category theory.

- Weighted limits and colimits.
- Pointwise extensions.
- Presheaves and the Yoneda lemma.
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What these concepts have in common is they rely in some way on the homs of a category. What structure does the hom-set of a small category form?

Answer: a distributor (a.k.a. profunctor, (bi)module).

To capture the structure of category theories, we must also consider distributors.

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Fortunately not all is lost. The composite of two $\mathbb{V}$-distributors is given by a colimit in $\mathbb{V}$. Hence, the data of a $\mathbb{V}$-natural transformation $q \odot p \Rightarrow r$ may be re-expressed without the assumption that $q \odot p$ exists. Axiomatising this situation leads to the notion of virtual double category.

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A pseudo double category is a generalisation of the notion of strict double category, in which we only require that the composition of loose-cells is associative and unital up to coherent isomorphism.

A virtual double category is a generalisation of the notion of pseudo double category in which we may not compose loose-cells at all. Accordingly, the notion of 2-cell must be generalised to have multiary domain.

$$
\begin{array}{ccccccccc}
A_n & \xrightarrow{p_n} & A_{n-1} & \xrightarrow{p_{n-1}} & \cdots & \xrightarrow{p_2} & A_1 & \xrightarrow{p_1} & A_0 \\
{\scriptstyle g}\downarrow & & & & \phi & & & & \downarrow{\scriptstyle f} \\
B_n & & & & \xrightarrow[q]{\hspace{6em}} & & & & B_0
\end{array}
$$

$\mathbb{V}$-forms

Let $\mathbb{V}$ be a monoidal category. A $\mathbb{V}$-form

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comprises a morphism

$$\phi_{x_0,\ldots,x_n} \colon p_1(x_0, x_1) \otimes \cdots \otimes p_n(x_{n-1}, x_n) \to q(fx_0, gx_n)$$

in $\mathbb{V}$ for each $x_0 \in |A_0|, \ldots, x_n \in |A_n|$, satisfying certain $\mathbb{V}$-naturality laws.

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$\mathbb{V}$-categories, $\mathbb{V}$-functors, $\mathbb{V}$-distributors, and $\mathbb{V}$-forms form a virtual double category $\mathbb{V}$-$\mathbf{Cat}$.

Equipments

The virtual double category $\mathbb{V}\text{-}\mathbf{Cat}$ is particularly well behaved.

1. For every $\mathbb{V}$-category A, there is a $\mathbb{V}$-distributor $A(-_1, -_2) \colon A \nrightarrow A$ sending $x, y \in |A|$ to $A(x, y)$. This satisfies a universal property making it the nullary composite of distributors on A.

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Virtual double categories satisfying these properties are called equipments, and are an appropriate setting for formal category theory [CS10].

The formal theory of relative monads

Relative monads and adjunctions in an equipment

The definitions of relative monad and relative adjunction generalise directly to the context of an equipment $\mathbb{X}$, by replacing

$$
\begin{aligned}
\text{categories} &\mapsto \text{objects in } \mathbb{X} \\
\text{functors} &\mapsto \text{tight-cells in } \mathbb{X} \\
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We then recover various notions of relative monad and relative adjunction by specialising to different equipments.

Examples 3

- A relative monad in $\mathbb{V}$-$\mathbf{Cat}$ is a $\mathbb{V}$-enriched relative monad.
- A relative monad in $\mathbf{Cat}(\mathbb{E})$ is an $\mathbb{E}$-internal relative monad.
- A relative monad in $\mathbb{V}$-$\mathbf{Act}$ is a $\mathbb{V}$-strong relative monad.

Basic theory

The basic theory of ordinary relative monads carries over without surprise to the setting of relative monads in equipments. For example:

Proposition 4

Every relative adjunction induces a relative monad.

Proposition 5

1. *Left j-relative adjoints preserve colimits preserved by j.*
2. *Right j-relative adjoints preserve limits when j is dense.*

Proposition 6

For a monad T on E, each tight-cell $j \colon A \to E$ induces a j-relative monad $j \,; T$ by precomposition.

Relative monads as monoids

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Relative monads may also be presented as monoids in hom-categories.

Theorem 7

Let $\mathbb{X}$ be an equipment. For each tight-cell $j \colon A \to E$, there is a skew-multicategory $\mathbb{X}[j]$ whose objects are tight-cells $A \to E$. Furthermore, monoids in $\mathbb{X}[j]$ are precisely j-relative monads.

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Theorem 8

If $\mathbb{X}$ furthermore admits left extensions of tight-cells $A \to E$ along $j \colon A \to E$, the skew-multicategory $\mathbb{X}[j]$ is representable by a skew-monoidal category.

Closing remarks

Monads in 2-categories vs double categories

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- Left adjoints preserve weighted colimits.
- Forgetful functors create weighted limits.
- Monadicity theorem.
- Algebras arise as a cocompletion of free algebras.

Algebra-objects in double categories

Definition 9

An algebra-object for a relative monad T is a universal T-algebra.

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Our notion of algebra-object is stronger than the traditional 2-categorical notion, even when T is a (non-relative) monad. The Eilenberg–Moore category for a (relative) monad in $\mathbb{V}$-$\mathbf{Cat}$ satisfies this stronger, double-categorical universal property.

In fact, this stronger universal property is necessary to establish some desirable properties of algebra-objects.

Proposition 10

An algebra-object $u_T \colon \mathbf{Alg}(T) \to E$ for a relative monad T creates limits and j-absolute colimits.

Summary

- Relative monads are generalisations of monads to arbitrary functors.
- Formal category theory is the study of category theory, using 2-dimensional category theory.
- 2-categories are an insufficient setting for many formal theorems about (relative) monads: we need the expressivity of double categories, or similar.

You can read our preprint on arXiv, where we develop much of the fundamental theory of relative monads in a formal setting, in particular specialising to $\mathbb{V}\text{-}\mathbf{Cat}$:

The formal theory of relative monads [AM23]

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